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LETTER TO THE EDITOR

Green function for charge–monopole scattering

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Abstract. We obtain the Green function for a non-relativistic charged particle moving in the field of a Dirac monopole. Several properties of the Green function are derived.

The Green function has held a time-honoured place in quantum theory. Recent developments have only served to verify its utility in many areas of physics (see, for instance, Gutzwiller *et al* 1986). It appears, however, that little attention has been given to applying the Green function to scattering off a monopole. In this letter we address ourselves to the non-relativistic scattering problem and show that the Green function can provide useful insights into monopole physics, a subject gaining more prominence in recent years.

Although a number of conclusions will immediately emerge from our results, perhaps the most important one is the derivation of a non-integrable phase factor. Over ten years ago a formulation of gauge fields in terms of non-integrable phase factors was advocated by Wu and Yang (1975). Many of their arguments were based on the classical notion of 'path'. We show here that for a charge-monopole system a quantum derivation of this factor can be given. This calculation gives us confidence that the phase factor really describes the transition amplitude for a charge moving about a monopole and not a flux about some hypothetical path. Classically such paths make sense, but in the quantum theory the concept of path gives way to that of the transition amplitude.

We begin by writing down the Schrödinger equation for a charge e of mass M in the field of an infinitely massive monopole of charge g:

$$\frac{1}{2M}\left(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A}\right)^2\psi = E\psi \tag{1}$$

where the vector potential $\boldsymbol{A}$ is given in spherical coordinates by

$$\boldsymbol{A} = \hat{\boldsymbol{\phi}}(g/r)\tan(\tfrac{1}{2}\theta) \tag{2}$$

with

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} = g\boldsymbol{r}/r^3.$$

The vector potential is singular along the z axis. By imposing the Dirac quantisation condition

$$eg/\hbar c = s = \text{integer or half-integer} \tag{3}$$

it can be shown that henceforth this string singularity plays no role in the physics of the system (Dirac 1931). The eigensolutions of this problem have been known for

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some time (Goldhaber 1965, Boulware *et al* 1976) and are given by

$$\begin{aligned}
&\psi_{Jm}(\boldsymbol{r}) = (2J+1)^{1/2} j_q(kr) d^{(J)}_{s-ms}(\theta) \exp(\mathrm{i}m\phi) \\
&J = |s|, |s|+1, |s|+2, \ldots \\
&m = -J, -J+1, \ldots, J-1, J \\
&k = (2ME)^{1/2} \qquad E > 0 \\
&q + \tfrac{1}{2} = [(J+\tfrac{1}{2})^2 - s^2]^{1/2}
\end{aligned} \tag{4}$$

where j_q is a spherical Bessel function, $d^{(J)}$ is a rotation function and s is defined by (3). There are no negative-energy solutions.

The Green function may now be written as an expansion in terms of the normalised eigensolutions:

$$\begin{aligned}
G(\boldsymbol{r}, t; \boldsymbol{r}', t') = &\sum_{Jm} (2J+1) \int \frac{\mathrm{d}^3 k}{(2\pi)^3} j_q(kr) j_q(kr') \\
&\times \exp\left(-\mathrm{i}\frac{k^2}{2M}(t-t')\right) d^{(J)}_{s-ms}(\theta) d^{(J)}_{s-ms}(\theta') \exp[\mathrm{i}m(\phi - \phi')].
\end{aligned} \tag{5}$$

Here we assume $t > t'$. The sum over m may be carried out following the usual rules of angular momentum algebra (Edmonds 1960):

$$\begin{aligned}
\sum_m d^{(J)}_{s-ms}(\theta) d^{(J)}_{s-ms}(\theta') \exp[\mathrm{i}m(\theta - \theta')] \\
&= \sum D^{(J)}_{ss-m}(\phi, -\theta, -\phi) D^{(J)}_{s-ms}(\phi', \theta', -\phi') \\
&= D^{(J)}_{ss}(\alpha, \gamma, \beta)
\end{aligned} \tag{6}$$

where (α, γ, β) are the Euler angles corresponding to the successive rotations $(\phi', \theta', -\phi')$ and $(\phi, -\theta, -\phi)$ and

$$\cos\gamma = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi' - \phi).$$

The k integral is not convergent. However, if we go over to Euclidean space $(t-t') \to -\mathrm{i}(t-t')$, then the integral may be evaluated using Weber's formula (Watson 1958). In Minkowski space then our Green function takes the form

$$\begin{aligned}
G(\boldsymbol{r}, t; \boldsymbol{r}, t') = &\frac{M}{\mathrm{i}T(rr')^{1/2}} \exp\left(\frac{\mathrm{i}M}{2T}(r^2 + r'^2)\right) \exp(\mathrm{i}s\Omega) \\
&\times \sum \frac{2J+1}{4\pi} J_\mu\left(\frac{Mrr'}{T}\right) d^{(J)}_{ss}(\gamma) \exp(-\mathrm{i}\mu\pi/2)
\end{aligned} \tag{7}$$

where $\mu = q + \frac{1}{2}$, $T = t - t'$ and $\Omega = \alpha + \beta$ is the solid angle subtended at the origin by the z axis and the vector $\boldsymbol{r}$ and $\boldsymbol{r}'$. We see above the phase factor $\exp(\mathrm{i}s\Omega)$. Several consequence may now be obtained.

(*a*) We may calculate the scattering amplitude as follows. The scattered wave $\psi^+_k(r)$ has the form

$$\psi^+_k(r) = \lim_{t' \to -\infty} \left(\frac{2\pi\mathrm{i}|t'|}{M}\right)^{3/2} G(\boldsymbol{r}, 0; \boldsymbol{r}', t') \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - k^2 t^2/2M)]$$

as shown by Pechukas (1969). Substituting the expression for G we have

$$\psi_k^+(r) \sim \frac{1}{2kr} \exp(\mathrm{i}s\Omega) \sum_J (2J+1) d_{ss}^{(J)}(\gamma)[\exp(\mathrm{i}kr) \exp(-\mathrm{i}\pi\mu) + \mathrm{i}\exp(-\mathrm{i}kr)]. \tag{8}$$

The second term represents an incoming wave so that the scattered wave is simply

$$\psi_{\text{scat}}(r) \sim (1/r) \exp(\mathrm{i}s\Omega) f(\gamma) \exp(\mathrm{i}kr) \tag{9}$$

where the scattering amplitude $f(\gamma)$ is given by

$$f(\gamma) = \frac{1}{2k} \sum_{J=s} (2J+1) d_{ss}^{(J)}(\gamma) \exp(-\mathrm{i}\pi\mu) \tag{10}$$

a result obtained by Boulware *et al* (1976).

(*b*) An interpretation for (7) can be arrived at by considering the Green function for infinitesimal times. Replacing T in (7) by ε we write it as ($s\delta\Omega$ is an infinitesimal flux)

$$G(\boldsymbol{r}, t; \boldsymbol{r}', t') = \frac{M}{\mathrm{i}\varepsilon(rr')^{1/2}} \exp\left(\mathrm{i}\frac{M}{2\varepsilon}(r^2 + r'^2) + \mathrm{i}s\delta\Omega\right) \sum_J \frac{2J+1}{4\pi} I_\mu\left[\frac{Mrr'}{\mathrm{i}\varepsilon}\right] d_{ss}^{(J)}(\gamma) \tag{11}$$

where I_μ is an associated Bessel function. The limit $\varepsilon \to 0$ may be obtained with the aid of the asymptotic expansion

$$I_a(x) \underset{x\to\infty}{\sim} \frac{1}{(2\pi x)^{1/2}} \exp\left(x - (a^2 - \tfrac{1}{4})\frac{1}{2x}\right). \tag{12}$$

The result is

$$\begin{aligned} G(\boldsymbol{r}t; \boldsymbol{r}'t') \underset{\varepsilon\to 0}{\sim} & \left(\frac{M}{2\pi\mathrm{i}\varepsilon}\right)^{1/2} \frac{1}{rr'} \exp\left(\mathrm{i}\frac{M}{2\varepsilon}(\boldsymbol{r} - \boldsymbol{r}')^2 + \mathrm{i}s\delta\Omega + \mathrm{i}\frac{\varepsilon s^2}{2Mrr'}\right) \\ & \times \sum_J \frac{2J+1}{4} d_{ss}^J(\gamma) \exp\left(-\mathrm{i}J(J+1)\frac{\varepsilon}{2Mrr'}\right). \end{aligned} \tag{13}$$

The sum over J may be evaluated by rewriting a formula given in appendix A of Boulware *et al* (1976)

$$\begin{aligned} \sum_J (2J+1) i_J(-\mathrm{i}kr) d_{ss}^J(\theta) = {} & \tfrac{1}{2}(\pi k\xi)^{1/2} \exp(\mathrm{i}k\eta/2) \\ & \times [\exp(-\mathrm{i}\pi/4) I_{s-1/2}(-\tfrac{1}{2}\mathrm{i}k\xi) - \mathrm{i} I_{s+1/2}(-\tfrac{1}{2}\mathrm{i}k\xi)] \end{aligned}$$

where $\xi = r(1+\cos\theta)$ and $\eta = r(1-\cos\theta)$. Applying expansion (12) on both sides of this result we obtain the desired expression:

$$\sum_J (2J+1) d_{ss}^J(\gamma) \exp\left(-\frac{\mathrm{i}}{2kr} J(J+1)\right) \underset{k\to 0}{\sim} -2\mathrm{i}kr \exp(\mathrm{i}k\eta - \mathrm{i}s^2/k\xi).$$

The Green function for infinitesimal times is then

$$G(\boldsymbol{r}t; \boldsymbol{r}'t') \underset{\varepsilon\to 0}{\sim} \left(\frac{M}{2\pi\mathrm{i}\varepsilon}\right)^{3/2} \exp\left(\frac{\mathrm{i}M}{2\varepsilon}(\boldsymbol{r} - \boldsymbol{r}')^2 + \mathrm{i}s\delta\Omega + \mathrm{O}(\varepsilon^{3/2})\right). \tag{14}$$

Now it is known that the infinitesimal propagator in a vector field $\boldsymbol{A}$ is given by (Schulman 1981)

$$\left(\frac{M}{2\pi\mathrm{i}\varepsilon}\right)^{3/2} \exp\left(\frac{\mathrm{i}M}{2\varepsilon}(\boldsymbol{r} - \boldsymbol{r}')^2 + \mathrm{i}e(\boldsymbol{r} - \boldsymbol{r}') \cdot \boldsymbol{A}(\tfrac{1}{2}\boldsymbol{r} + \tfrac{1}{2}\boldsymbol{r}')\right). \tag{15}$$

Comparing the last two results we conclude that the flux term $s\delta\Omega$ in (14) comes from the line integral of $\boldsymbol{A}$. This line integral may be interpreted as an infinitesimal loop integral by completing the loop with circular arcs to the z axis. Since these arcs intersect $\boldsymbol{A}$ at right angles they do not contribute to the loop integral. Moreover, for the finite-time Green function the line integral of $\boldsymbol{A}$ must be responsible for the flux term $s\Omega$ in (7), although it is clear that it also contributes to the other factors in the Green function. Nevertheless the line integral of $\boldsymbol{A}$ cannot be looked upon as a simple loop integral since the quantum propagator involves a sum over paths. Thus the non-integrable flux factor hides a 'sum-over-loop integral' of the vector field. Note that the phase factor for an infinitesimal path is just a flux; for a finite path the amplitudes sum up in a complicated way although a flux still emerges.

(*c*) We show that $s=$ half-integer (3). This is the Dirac quantisation condition. Consider the amplitude for the charge to make a complete circuit about the string: $(r, \theta, \phi) \rightarrow (r', \theta, \phi+2\pi)$. According to the above discussion the Green function is proportional to $\exp(\mathrm{i}s\Omega)$ where Ω is the solid angle subtended by a circular cap around the z axis with polar angle θ. Since the description of the system cannot depend on the position of the string we expect the Green function to remain unchanged if the string were rotated onto the positive z direction. In this case the Green function would be proportional to $\exp(-\mathrm{i}s\Omega')$ where Ω' is the solid angle of the same cap as viewed from the negative z direction. The minus sign is due to the direction of motion of the charge. We have then

$$\exp(\mathrm{i}s\Omega)=\exp(-\mathrm{i}s\Omega')$$

or

$$\exp(\mathrm{i}4\pi s)=1.$$

It follows that

$$4\pi s=2\pi n \qquad n=0,1,2,\ldots \tag{16}$$

which is the result of Dirac (1931). Our discussion does not make use of classical paths around the string.

The result just obtained allows us to speak of gauge transformation. A shift of the string from the negative to the positive z direction is really a gauge transformation. Thus the Dirac condition can be looked upon as a statement of the invariance of the phase factor around a loop under a gauge transformation. The special choice of the z axis for the string does not prevent us from selecting other directions. It follows that the invariance of the phase factor is true for any gauge transformation provided the Dirac condition is satisfied. We may now follow up this discussion with that of Wu and Yang (1975), but this time, however, without having to use the idea of a classical path about the monopole.

(*d*) We may extend our discussion to dyon–dyon scattering. The non-relativistic problem has been solved by Schwinger *et al* (1976). In terms of the reduced mass μ and the invariant charge combinations

$$\begin{aligned} S&=-(e_1g_2-e_2g_1)/\hbar c \\ Q&=-2\mu(e_1e_2+g_1g_2)/\hbar^2 \end{aligned} \tag{17}$$

they find that the eigensolutions are given by

$$(2J+1)^{1/2}\exp(\mathrm{i}m\phi)\, d^{(J)}_{m-SS}(\theta)R_k(r) \tag{18}$$

where $R_k(r)$ is a radial function satisfying the radial Schrödinger equation for Coulomb scattering:

$$\left(\frac{d^2}{ds^2}+\frac{2}{r}\frac{d}{dr}+k^2+\frac{2}{r}-\frac{J(J+1)-S^2}{r^2}\right)R_k(r)=0 \qquad J=|m-S|, |m-S|+1, \ldots .$$

Both bound and scattering solutions exist. When (18) is substituted into the expansion for the Green function we obtain an expression of the form

$$\exp(\mathrm{i}s\Omega)\sum_J \frac{2J+1}{4\pi}\, d_{SS}^{(J)}(\gamma) \text{ (radial factor).} \tag{19}$$

We have not been able to obtain a closed form for the radial factor but it is just what one would obtain for pure Coulomb scattering, except that J begins with S, and not 0.

Two results follow immediately. As in (16), we have

$$S = \text{half-integer or zero.}$$

Also when $S=0$, i.e. scattering of identical dyons, the Green function is formally identical to that for pure Coulomb scattering since $d_{00}^{(J)}=P_J$. In this special case the sum may be evaluated following the method of Hostler (1964).

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